Analytical solutions of Fokker-Planck equation

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(Dated: 22 October 2020)

Some physical, chemical and biological processes which are stochastical are governed by Fokker-Planck equation. The solution gives the distribution function of particles undergoing different processes (Diffusion, Brownian motion etc.,). In this paper, the analytical solutions for simple problems is solved by treating one dimensional Fokker-Planck equation.

I. INTRODUCTION

The Fokker-Planck equation gives the time evolution of distribution function of a particle undergoing a diffusional motion in one dimension is given as,

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2} D^{(2)}(x)P - \frac{\partial}{\partial x} D^{(1)}(x)P - k(x)P \quad (1)$$

where $D^{(2)}(x)$ and $D^{(1)}(x)$ are called diffusion and drift coefficients respectively and $k(x)$ is the sink term which represents the decay of the population of distribution.
The drift coefficient for the particles $D^{(1)}(x) = 0$ and the sink term $k(x) = 0$ signifies that the particles density is conserved and they just undergo the diffusion process (Wiener process, otherwise). If the drift coefficient for the particles $D^{(1)}(x)$ is linear in space and has a constant diffusion coefficient the process becomes Ornstein-Uhlenbeck process. If the potential is parabolic then the diffusion is from a harmonic potential well. The paper is concerned with the analytical solving of the resulting partial differential equations for these processes using various mathematical techniques.

II. WEINER PROCESS

A stochastic process with vanishing drift coefficient ($D^{(1)}(x) = 0$) and constant diffusion coefficient is called a Weiner process. Hence becomes an Diffusion equation,

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P}{\partial x^2}$$

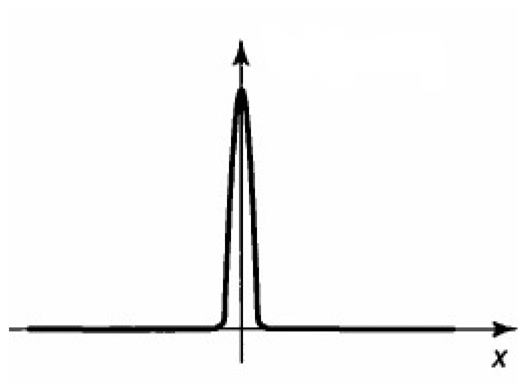
Weiner process serves as a dynamical model for the diffusion of a Brownian particle and to represent the Gaussian white noise in electrical circuits. This equation can also be heat diffusion equation with D as heat diffusivity.
Taking Laplace transform of the diffusion equation,

$$\hat{L}\frac{\partial P(x,t)}{\partial t} = D\hat{L}\frac{\partial^2 P}{\partial x^2}$$

$$P(x,0) + s\tilde{P}(x,s) = D\frac{\partial^2 \tilde{P}(x,s)}{\partial x^2}$$

For a diffusion process, $P(x,0) = \delta(x)$

$$s\tilde{P}(x,s) + \delta(x) = D\frac{\partial^2 \tilde{P}(x,s)}{\partial x^2}$$



The solution in region $x \neq 0$,

$$\tilde{P}(x,s) = \begin{cases} Ae^{-kx} & x > 0 \\ Be^{kx} & x < 0 \end{cases} \qquad \text{where} \quad k = \sqrt{\tfrac{s}{D}}$$

Encountering continuity property,

$$\tilde{P}(x,s)_+ = \tilde{P}(x,s)_-$$

$$\left(\frac{\partial \tilde{P}(x,s)}{\partial x}\right)_+ - \left(\frac{\partial \tilde{P}(x,s)}{\partial x}\right)_- = 1/D$$

$$\tilde{P}(x,s) = \frac{1}{2\sqrt{sD}} \begin{cases} e^{-\sqrt{\frac{s}{D}}x} & x > 0 \\ e^{\sqrt{\frac{s}{D}}x} & x < 0 \end{cases}$$

$$P(x,t) = C\frac{e^{-x^2/4Dt}}{\sqrt{2Dt}}$$

Inverse Laplace transform is not solvable in many cases[3]

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P}{\partial x^2}$$

The PDE is now turned into an ODE by Fourier tranform. The solution is assumed to be of Fourier integral form

$$\int_{-\infty}^{\infty} e^{-i\omega x}\frac{\partial \tilde{P}(\omega,t)}{\partial t}d\omega = D\int_{-\infty}^{\infty} e^{-i\omega x}(-\omega^2)\tilde{P}(\omega,t)d\omega$$

$$\frac{\partial \tilde{P}(\omega,t)}{\partial t} = D(-\omega^2)\tilde{P}(\omega,t)$$

$$\tilde{P}(\omega, t) = Ce^{-D\omega^2 t}$$

Doing Fourier transform,

$$P(x,t) = C\frac{e^{-x^2/4Dt}}{\sqrt{2Dt}}$$

The solution gives that the probability distribution is of Gaussian form with zero mean and second moment linear in time as of for the processes such as Gaussian white noise, random walk etc..

III. DIFFUSION FROM A LINEAR POTENTIAL

Diffusion from a linear potential well is a model for many chemical process[5], The Smoluchowski equation for such a case reads,

$$\frac{\partial P(x,t)}{\partial t} = D\left(\frac{\partial^2}{\partial x^2}P - \frac{\partial}{\partial x}P\frac{\partial U}{\partial x}\right) - k(x)P$$

The potential is linear ($U(x) = -kx$) and sink $k(x) = k_0\delta(x - x_s)$. Taking Laplace transform and solving ODE,

$$\hat{L}\frac{\partial P(x,t)}{\partial t} = D\left(\hat{L}\frac{\partial^2}{\partial x^2}P - \hat{L}\frac{\partial}{\partial x}P\frac{\partial U}{\partial x} - k_0\delta(x - x_s)P(x,t)\right)$$

$$s\tilde{P}(x,s) = D\frac{\partial^2\tilde{P}(x,s)}{\partial x^2} + \delta(x) + Dk\frac{\partial\tilde{P}}{\partial x} - k_0\delta(x - x_s)\tilde{P}(x,s)$$

$$\tilde{P}(x,s) = Ae^{(p-q)x} + Be^{-(p+q)x}$$

where $q = k/2$ and $p = \sqrt{q^2 + \frac{s}{D}}$. Inverse Laplace transform was attempted by Privman and Frish[3] for a similar problem with $U(x) = k|x|$. But they could obtain only high barrier limit through a more involved procedure.

IV. SUMMARY

As many of the real life problems can be mapped into Fokker-Planck equation we are trying to give exact analytical solutions for those equations. Our aim is to develop new mathematical techniques to overcome problems in getting exact analytical solutions (such as problems in taking Inverse Laplace transform etc.,). The problem is viewed in an different aspect and we are trying to develop an solution in the time domain itself[6].

V. ACKNOWLEDGEMENT

Thanks to Prof. Dr. Aniruddha Chakraborty who gave useful discussions and motivating throughout my work.

Thanks to my friend Chinmoy Samanta who joyfully took part in discussions.

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